Growth and Structure of Stochastic Sequences

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We introduce a class of stochastic integer sequences. In these sequences, every element is a sum of two previous elements, at least one of which is chosen randomly. The interplay between randomness and memory underlying these sequences leads to a wide variety of behaviors ranging from stretched exponential to log-normal to algebraic growth. Interestingly, the set of all possible sequence values has an intricate structure.

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Integer sequences underly many problems in combinatorics, computer science, and physics, with new beautiful sequences continuing to emerge [1]. Sequences are typically deterministic. Meanwhile, stochastic sequences are just as ubiquitous, occurring in random processes such as the random walk. Stochastic sequences usually arise in very different contexts, and hence are rarely compared with their deterministic counterparts. In this article, we demonstrate how rich such a comparison can be.

Consider the Fibonacci numbers, $F_n = F_{n-1} + F_{n-2}$, that describe for example the number of leaves in plants and the number of ancestors of a drone [2, 3, 4]. As every element depends on the previous two, a natural stochastic generalization is $x_n = x_{n-1} \pm x_{n-2}$, where addition and subtraction are chosen with equal probabilities [5, 6, 7, 8] (similar sequences also describe one-dimensional disordered systems [9, 10]). The resulting sequences are intriguing. While the sequences still grow exponentially, the ratio x_n/x_{n-1} approaches a stationary distribution that possesses singularities at all rational values [7, 8].

Inspired by this richness, we consider an alternative form of stochasticity, namely, one that does *not* require subtraction and therefore more similar in spirit to the original deterministic sequence. Relaxing the rule that every element depends only on the preceding two elements, we arrive at the following additive stochastic rules

$$x_n = \begin{cases} x_{n-1} + x_q & \text{(model I);} \\ x_p + x_q & \text{(model II).} \end{cases} \tag{1}$$

In model I, we take the preceding element x_{n-1} and another one whose index q is randomly chosen between 0 and $n-1$. In model II, both indices p and q are chosen randomly, $0 \le p, q \le n-1$. Without loss of generality, the first element is set to unity, $x_0 = 1$. Consequently $x_1 = 2$, while the next elements are stochastic. The number of possible sequences increases as $n!$ and $n!^2$ for models I and II, respectively. Rule I leads to monotonically increasing sequences; sequences generated by rule II increase only on average.

The most basic characteristic, the average of the nth element, $A_n = \langle x_n \rangle$, can be determined analytically. For model I, it satisfies the linear recursion relation

$$A_n = A_{n-1} + \frac{1}{n}\sum_{j=0}^{n-1} A_j. \tag{2}$$

Comparing this with the recursion for A_{n+1}, we eliminate the summation and obtain a Fibonacci-like recursion relation with n-dependent coefficients $A_{n+1} - 2A_n + A_{n-1} = (n+1)^{-1}A_{n-1}$. We are primarily interested in the large-n behavior, and hence, we treat A and n as continuous variables. The above difference equation reduces to the differential equation $A'' = (A - A')/n$ (here $' \equiv d/dn$). Using the WKB method [11], we obtain the n-dependence

$$A_n \simeq a n^{-1/4} \exp\left(2\sqrt{n}\right). \tag{3}$$

The amplitude $a \approx 0.1711$ is determined numerically. We see that the long range memory leads to considerably slower stretched exponential growth compared with the exponentially growing Fibonacci numbers.

Does the average characterize the growth of an actual sequence? If yes, this would imply that the normalized moments $\langle x_n^k \rangle / \langle x_n \rangle^k$ approach finite values asymptotically. Figure 1 shows otherwise: the higher order moments grow according to

$$\langle x_n^k \rangle \propto \exp\left(\beta_k \sqrt{n}\right), \tag{4}$$

with $\beta_k > k\beta_1$ (for the lowest moments, we find $\beta_k = 2$, 4.3, and 6.5). This so-called "multiscaling" indicates that a typical sequence may greatly depart from the average. Therefore, the average (3) is insufficient to describe a typical sequence. These results are similar in spirit to the behavior of the random Fibonacci sequences where the typical growth is $x \propto \exp(\gamma n)$, with nontrivial Lyapunov exponent γ, while $\langle x_n^k \rangle \propto \exp(\gamma_k n)$ with $\gamma_k \neq k\gamma_1$.

Interestingly, an individual realization grows slower than the average (see the inset to Fig. 1)

$$x_n \propto \exp\left(\beta\sqrt{n}\right), \tag{5}$$

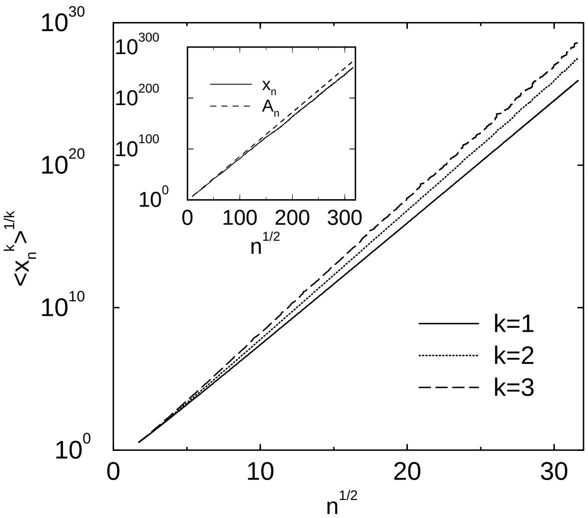

FIG. 1: The moments $\langle x_n^k \rangle^{1/k}$ versus n for model I. The moments were obtained from an average over 10^8 realizations. The inset compares the growth of an individual realization x_n with the average A_n.

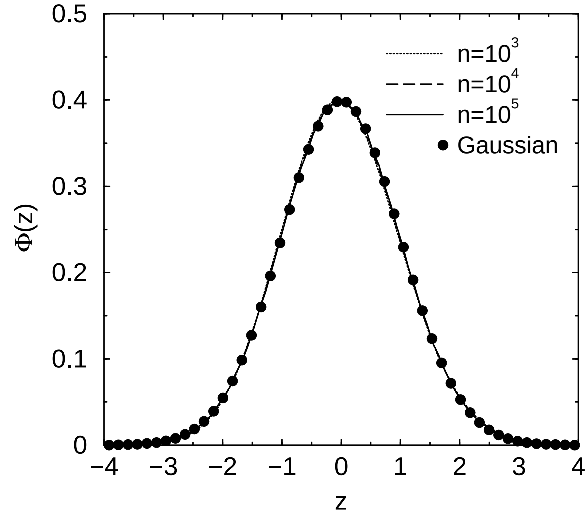

FIG. 2: The scaling distribution underlying $P_n(\ln x_n)$. Shown is the scaling function $\Phi(z)$ versus the scaling variable $z = (\ln x_n - b_n)/\Delta_n$. The distributions were calculated from 10^7 realizations. A Gaussian is also shown for reference.

with the Lyapunov exponent $\beta \approx 1.889$. This coefficient was determined by studying the variable $\ln x_n$. As shown in Fig. 2, this variable is Gaussian distributed

$$P_n(\ln x_n) \propto \exp\left[-\frac{(\ln x_n - b_n)^2}{2\Delta_n^2}\right]. \tag{6}$$

The average and the variance of $\ln x_n$ grow with n according to $b_n \simeq \beta n^{1/2}$ and $\Delta_n^2 \simeq \sigma^2 n^{1/2} \propto b_n$, respectively. Eventually, the random variable $y = \ln x_n / n^{1/2}$ becomes deterministic, $y \to \beta$ as $n \to \infty$. Similar behavior, including the Gaussian fluctuations and the relation between the variance and the average, is also found in one-dimensional localization problems [12, 13].

One can calculate the probability distribution $P_n(x)$ for extremal values of x_n. The minimal value $n+1$ is obtained by choosing $q = 0$ at every step $k = 1, \ldots, n$. Similarly, the maximal value 2^n is obtained by choosing $q = k-1$ at every step. Hence, $P_n(n+1) = P_n(2^n) = 1/n!$. Further extremal cases can be evaluated manually, and for example, $P_n(n+2) = P_n\left(3 \cdot 2^{n-2}\right) = \frac{n-1}{n!}$. However, these extremal probabilities do not elucidate the typical behavior (5).

It is interesting to study the set of all possible sequence values. Let Ω_n be the support of the probability distribution $P_n(x)$. For small n, we have $\Omega_0 = \{1\}$, $\Omega_1 = \{2\}$, and furthermore,

$$\begin{aligned}
\Omega_2 &= \{3, 4\} \\
\Omega_3 &= \{4, 5, 6, *, 8\} \\
\Omega_4 &= \{5, \ldots, 10, *, 12, *, *, *, 16\} \\
\Omega_5 &= \{6, \ldots, 18, *, 20, *, *, *, 24, *, \ldots, *, 32\}.
\end{aligned} \tag{7}$$

Determination of the sets Ω_n requires enumeration of all $n!$ histories, and we computed them up to $n = 15$. The simplest feature is the set size Γ_n, listed in table I. The set Ω_n always begins with a subsequence of B_n consecutive integers. For $n \geq 3$, the sets Ω_n have gaps, i.e., strings of missing elements denoted by $*$ in Eq. (7). The number of gaps G_n is listed in table I. The three sequences Γ_n, B_n, and G_n, all grow exponentially with n. For example, $\Gamma_n \propto \lambda^n$ with $\lambda \approx 1.78$. We conclude that the sets Ω_n contain a number of nontrivial deterministic integer sequences including Γ_n, B_n, and G_n.

Remarkably, the sets Ω_n have an intricate structure. For instance, for $n = 8$ the sequence of the gap lengths contains $G_8 = 18$ elements as follows

$$\{1, 1, 1, 1, 1, 3, 1, 1, 3, 3, 7, 7, 1, 3, 7, 15, 31, 63\}. \tag{8}$$

Generally, going in reverse direction (from 2^n to $n + B_n$) one observes a family of consecutive gaps of lengths $2^{n-2} - 1, 2^{n-3} - 1, \ldots, 1$ separated by single elements. Then, there is a 3-element sequence, followed by a second family of *twin* gaps, $2^{n-5} - 1, 2^{n-5} - 1, \ldots, 1, 1$. All these gaps are separated by single elements. Next, there is a 5-sequence, followed by a family of *triplet* gaps, again separated by single elements [this family has not yet formed for $n = 8$, Eq. (8)]. There is also a fourth family of *quadruplet* gaps with an intertwined pattern. The complexity of this gap-sequence structure increases rapidly, and eventually, gaps of even length appear.

Naively, one may probe the probability distribution via a mean-field description that ignores the sequence history altogether. In this approximation, one obtains a recursive equation for the probability distribution

$$P_n(x) = \frac{1}{n} \sum_{l=0}^{n-1} \sum_{y=1}^{x-1} P_{n-1}(y)\, P_l(x - y). \tag{9}$$

n	Γ_n	B_n	G_n
3	4	3	1
4	8	6	2
5	16	13	3
6	30	22	6
7	55	39	10
8	98	62	18
9	175	117	28
10	310	180	50
11	555	367	79
12	986	594	144
13	1757	1073	249
14	3138	1888	432
15	5618	3567	756

TABLE I: The sequences Γ_n, B_n, and G_n.

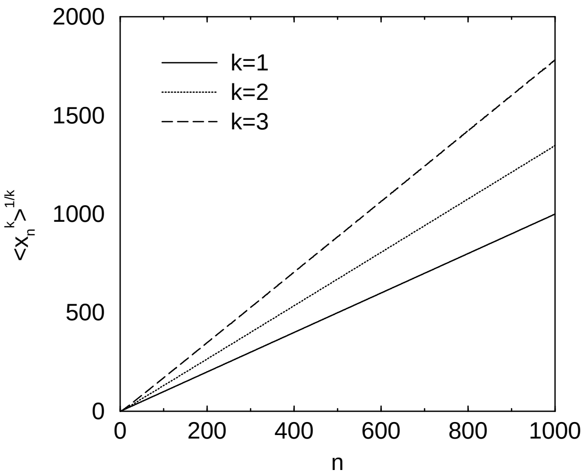

FIG. 3: The moments $\langle x_n^k \rangle^{1/k}$ versus n for model II. The data represents an average over 10^8 realizations.

Consequently, there are closed recursion relations for the moments. While consistent with the exact equation (2), the emerging recursion relations for the higher moments are only approximate. Analysis of these equations results in ordinary scaling behavior, $\langle x_n^k \rangle \propto \langle x_n \rangle^k$, contrary to Eq. (4). Therefore, strong correlations develop, correlations that affect the statistical characteristics.

We now turn to model II. Here, the average A_n satisfies a recursion relation similar to Eq. (2),

$$A_n = \frac{2}{n} \sum_{j=0}^{n-1} A_j. \tag{10}$$

Simplifying this equation to $A_n = \left(1 + n^{-1}\right) A_{n-1}$ and using the initial condition $A_0 = 1$, we obtain

$$A_n = n + 1. \tag{11}$$

Numerical simulations confirm this linear growth. Figure 3 shows that the normalized moments remain finite asymptotically as

$$\langle x_n^k \rangle \simeq \mu_k n^k. \tag{12}$$

In contrast with model I, the average properly characterizes higher order moments. Therefore, the probability distribution $P_n(x)$ admits the scaling form

$$P_n(x) \simeq n^{-1}\Phi(z), \qquad z = xn^{-1}, \tag{13}$$

in the asymptotic limit $x, n \to \infty$ with the scaling variable $z = xn^{-1}$ fixed (see Fig. 4).

This scaling behavior enables quantitative characterization of extremal statistics. Numerically, we find that the scaling function exhibits the following extremal behaviors

$$\Phi(z) \propto \begin{cases} z & z \to 0, \\ \exp\left(-z^\kappa\right) & z \to \infty, \end{cases} \tag{14}$$

with $\kappa \approx 0.4 - 0.5$ (see the inset to Fig. 4). The small-z behavior can be understood by considering the minimal value $x_n = 2$. This occurs when the first element $x_0 = 1$ is chosen twice and therefore, $P_n(2) = n^{-2}$. Combining this with Eq. (13), gives $\Phi(2n^{-1}) = n^{-1}$, in agreement with the asymptotics $\Phi(z) \propto z$ in the $z \to 0$ limit. The large-z behavior is more subtle as it depends on the entire sequence evolution. Contrary to the small argument behavior, analysis of the maximal sequences $x_n = 2^n$ does not elucidate the large argument tail. Indeed, such sequences occur with probability $1/n!$, much smaller than the exponentially small probabilities dominating the large-z behavior.

The ordinary scaling behavior indicates that mean-field theory may provide better insight in the case of model II. Ignoring the history by which sequences evolve yields the following recursion for the distribution

$$P_n(x) = n^{-2} \sum_{l=0}^{n-1} \sum_{m=0}^{n-1} \sum_{y=1}^{x-1} P_l(y)\, P_m(x-y). \tag{15}$$

Using Eq. (13), and replacing summations by integration, equation (15) reduces into the integral equation

$$\Phi(z) = \int_0^1 \frac{d\xi}{\xi} \int_0^1 \frac{d\eta}{\eta} \int_0^z dz'\, \Phi\left(\frac{z'}{\xi}\right) \Phi\left(\frac{z-z'}{\eta}\right). \tag{16}$$

The convolution structure suggests using the Laplace transform, and indeed, $F(s) = \int dz e^{-sz}\Phi(z)$ satisfies a simple equation

$$F(s) = \left[\int_0^1 d\xi\, F(\xi s)\right]^2. \tag{17}$$

The auxiliary function $G(s) = \int_0^s ds'\, F(s')$ obeys the ordinary differential equation $dG/ds = (G/s)^2$ from which $G(s) = s/(1+cs)$ and then $F(s) = (1+cs)^{-2}$. The small-s behavior $F(s) = 1 - s$ implies $c = 1/2$. Inverting the Laplace transform $F(s) = (1 + s/2)^{-2}$ yields the scaling function

$$\Phi(z) = 4z \exp\left(-2z\right). \tag{18}$$

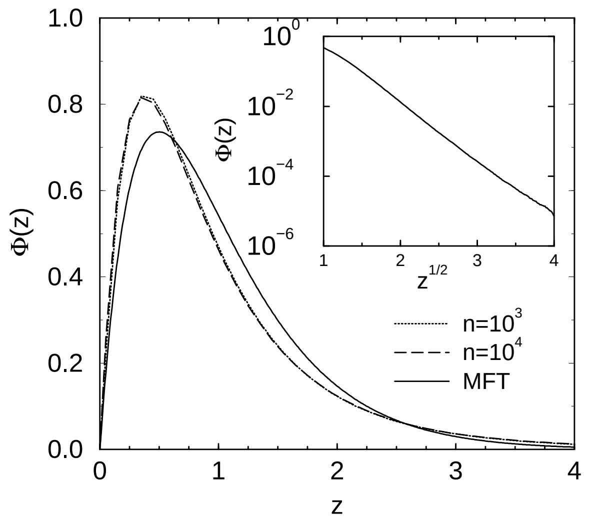

FIG. 4: The scaling distribution $\Phi(z)$ vs. z. The distributions were obtained from an average over 10^7 realizations. Shown also is the mean field theory (MFT) result (18). The inset shows the large argument tail.

In the small-z limit, mean-field theory is correct because memory is irrelevant for $x \ll n$, (see Fig. 3). In contrast, for $x \gg n$, memory is important and the exponent $\kappa = 1$ is larger than the numerical value $\kappa \approx 0.4-0.5$. Additionally, one may compare the prefactors characterizing the moments defined in Eq. (12). Using $\mu_k = \int dz z^k \, \Phi(z)$ gives $\mu_k = (k+1)!2^{-k}$ and in particular $\mu_k = 1,\ 3/2,\ 3$, for $k = 1,\ 2,\ 3$. The corresponding numerical values are 1, 1.84, 5.76, respectively. Although the history independent approximation is quantitatively inaccurate, it still provides useful insights for model II.

We have seen that the sequence growth sensitively depends on the details of the model. In fact, the stretched exponential or algebraic growth can be tuned by varying the recurrence rules. For example, introducing a multiplicative factor to model II, $x_n = c(x_p + x_q)$, leads to the algebraic growth $A_n \sim n^{2c-1}$. Functionally different growth laws naturally emerge as well. If in model I, $x_n = x_{n-1} + x_q$, the memory range is $0 \le q \le [bn]$ with $0 < b < 1$, one finds log-normal growth

$$A_n \propto \exp\left(C \ln^2 n\right), \tag{19}$$

with $C = [2\ln(1/b)]^{-1}$. This growth law is slower than stretched exponential but faster than power law.

In summary, we have introduced a class of stochastic integer sequences where each sequence element is a sum of two previous elements, at least one of which is randomly chosen. While the sequence may attain a vast range of possible values, the dynamics chooses a much narrower range of values. Depending on the governing rules, there is a wide spectrum of growth from algebraic to log-normal to stretched exponential. In model I, there are infinitely many relevant scales underlying the moments. In contrast, for model II, there is a single scale and consequently, a mean-field approximation is qualitatively correct.

Generally, the phase space has an intricate structure. It contains alternating sequences of consecutive integers marking allowed and forbidden sequence values. The gap structure consists of increasingly complex patterns. Interesting deterministic integer sequences such as the size of the phase space, the number of gaps, and the size of the first accessible sequence underlie this phase space.

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